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CHARACTERIZATION THEOREMS FOR INSURER EQUIVALENT UTILITY PREMIUM CALCULATION PRINCIPLE

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Резюме. В роботі представлено ряд характеризаційних теорем для декількох бажаних властивостей, якими може володіти або не володіти принцип еквівалентної корисності страховика підрахунку вартості страхових контрактів. Представлені теореми охоплюють властивості адитивності, консистентності, ітеративності та мультиплікативної інваріантності. Результати сформульовані у вигляді необхідних та достатніх умов володіння згаданими властивостями накладених на функцію корисності страховика. Характеризаційні твердження для принципу нульової корисності страховика сформульовані у вигляді наслідків до відповідних тверджень для принципу еквівалентної корисності страховика.

Резюме. В работе представлен ряд характеризационных теорем для нескольких желаемых свойств, которыми может обладать или не обладать принцип эквивалентной полезности страховщика подсчета стоимости страховых контрактов. Представленные теоремы охватывают свойства аддитивности, консистентности, итеративности и мультипликативной инвариантности. Результаты сформулированы в виде необходимых и достаточных условий выполнения упомянутых свойств наложенных на функцию полезности страховщика. Характеризационные утверждения для принципа нулевой полезности страховщика сформулированы в виде следствий к соответствующим утверждениям для принципа эквивалентной полезности страховщика.

Abstract. Characterization theorems for several desirable properties that can be possessed or not possessed by the insurer equivalent utility premium calculation principle are presented. Demonstrated theorems cover cases of additivity, consistency, iterativity, and scale invariance properties. Results are formulated in a form of necessary and sufficient conditions for attainment of the properties imposed on the insurer's utility function. Characterizations for the insurer zero utility principle are formulated as corollaries of corresponding characterizations for the insurer equivalent utility principle.

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Ключові слова: характеризаційна теорема, страхова премія, принцип еквівалентної корисності страховика, принцип нульової корисності страховика, властивість адитивності, властивість конзестентності, властивість ітеративності, властивість мультіплікативної інваріантності.

Ключевые слова: характеризационная теорема, страховая премия, принцип эквивалентной полезности страховщика, принцип нулевой полезности страховщика, свойство аддитивности, свойство конзестентности, свойство итеративности, свойство мультипликативной инвариантности.

Key words: characterization theorem, insurance premium, insurer equivalent utility premium principle, insurer zero utility premium principle, additivity property, consistency property, iterativity property, scale invariance property.

1. Introduction

Let us consider a random variable X representing size of the insurance compensation related to a particular insurance pact. Premium to be paid for the risk X will be denoted as $\pi[X]$.

In majority of the cases the random variable X is assumed to be a non-negative one, i.e., it takes vale zero if the contract will not produce a claim and will be equal to the claim size if there will be a claim. In some case, however, negative values of the variable X are also aloud; such negative values are often interpreted as compensations which have to be paid by the customer to the insurance company, for example, as penalties for violation of the contract conditions.

Let us now define several insurance premium calculation principles which we would like to investigate.

Insurer equivalent utility premium for a risk X, which we denote as $\pi_{\text{i.e.u.}}[X]$, is defined as a solution to the equation

$$U(W) = \mathsf{E}[U(W + \pi_{\text{i.e.u.}}[X] - X)], \quad (1)$$

where W is the insurer's capital at the moment when the contract is initiated, and the function $U(x) \in C_2(\mathbb{R})$ is the insurer's utility function, i.e., it satisfies conditions $U'(x) > 0$ and $U''(x) \le 0$ for $x \in \mathbb{R}$.

In some of the cases the insurer's utility function is selected in such a way that the value $U(0)$ represents insurer's utility at the moment when the contract is initiated. In such cases equation (1) for the risk X is replaced by the equation

$$U(0) = \mathsf{E}[U(\pi[X] - X)] \quad (2)$$

and corresponding method of pricing of the insurance contracts is called *insurer zero utility premium calculation principle*. Obtained in such a way premium in the article will be denoted as $\pi_{\text{i.z.u.}}[X]$.

Sometimes the insurer equivalent utility premium calculation principle and the insurer zero utility premium calculation principle are applied to some special classes of risks; as an example of such a class one can mention the class of all non-negative risks, alternatively one could mention the class of all non-negative risks bounded from above by some fixed real value, etc. In such cases the domain of the function $U(x)$ could be a subset of $\mathbb{R}$ such that the equation (1) or, alternatively, equation (2), depending on chosen method of pricing, will preserve its correct mathematical meaning, moreover, monotonicity and concavity properties of the function $U(x)$ should also be preserved.

Net premium for a risk X, which in the article will be denoted as $\pi_{\text{net}}[X]$, is defined as the expected value of the losses associated with the risk X, i.e.,

$$\pi_{\text{net}}[X] = \mathsf{E}[X].$$

Exponential premium, dependent on a parameter β, for a risk X which in the article will be denoted as $\pi_{\exp(\beta)}[X]$, is defined in the following way

$$\pi_{\exp(\beta)}[X] = \frac{1}{\beta}\log\left(\mathsf{E}[e^{\beta X}]\right), \quad \text{for} \quad \beta > 0.$$

We will say that a premium calculation principle $\pi[X]$ possesses:

additivity property if for any two independent risks X_1 and X_2 holds the identity

$$\pi[X_1 + X_2] = \pi[X_1] + \pi[X_2]; \quad (3)$$

consistency property if for any risk X and any real constant c (if a pricing method is defined only for the non-negative risks then the constant c can be claimed to be non-negative in order to avoid situations when $X + c < 0$, i.e., situations when the value $\pi[X + c]$ is undefined) holds the identity

$$\pi[X + c] = \pi[X] + c; \quad (4)$$

iterativity property if for any two risks X and Y holds the identity

$$\pi[\pi[X \mid Y]] = \pi[X]; \quad (5)$$

scale invariance property if for any risk X and any positive real constant Θ holds the identity

$$\pi[\Theta X] = \Theta\pi[X]. \quad (6)$$

More information about the defined methods of pricing of the insurance contracts as well as the properties that can be possessed by the insurance premium calculation principles can be found, for example, in Asmussen and Albrecher (2010), Boland (2007), Bowers et al (1997), Buhlmann (1970), Dickson (2005), Gerber (1979), De Vylder et al (1984), De Vylder et al (1986), Kaas et al (2008), Kremer (1999), Rolski et al (1998), Straub (1988).

We would like to emphasize that the research related to theorems of characterization type for properties possessed by certain insurance premium calculation principles was initiated by the Swiss mathematician Hans-Ulrich Gerber, see Gerber (1979).

2. Additivity Property

The following theorem describes the necessary and sufficient conditions for attainment of the additivity property by the insurer equivalent utility premium calculation principle.

Theorem 2.1. *The insurer equivalent utility premium calculation principle possesses the additivity property if and only if* $U(x) = ax + b,$ *for* $a > 0,$ *or* $U(x) = -\alpha e^{-\beta x} + \gamma,$ *for* $\min[\alpha, \beta] > 0,$ *i.e., only in the cases when it coincides with either the net premium principle or the exponential premium principle.*

Observe that the class of functions $U(x) = -\alpha e^{-\beta x} + \gamma,$ for $\min[\alpha,\beta] > 0,$ contains all functions of the form $U(x) = -\tau^{-x},$ for some real constant $\tau > 1.$

Proof. Let us from the beginning prove the sufficiency of the statement. We start from the case of $U(x) = ax + b,$ for $a > 0.$ Indeed, in this case for any two independent risks X_1 and X_2, and any insurer's initial capital $W,$ from the equation (1) it follows

$$aW + b = \mathsf{E}[a(W + \pi_{\text{i.e.u.}}[X_i] - X_i) + b], \quad \text{for} \quad i = \overline{1,2},$$

and thus

$$\pi_{\text{i.e.u.}}[X_i] = \mathsf{E}[X_i] = \pi_{\text{net}}[X_i], \quad \text{for} \quad i = \overline{1,2}.$$

On the other hand, from the same equation it follows

$$aW + b = \mathsf{E}[a(W + \pi_{\text{i.e.u.}}[X_1 + X_2] - X_1 - X_2) + b],$$

hence

$$\pi_{\text{i.e.u.}}[X_1 + X_2] = \mathsf{E}[X_1] + \mathsf{E}[X_2] = \pi_{\text{i.e.u.}}[X_1] + \pi_{\text{i.e.u.}}[X_2],$$

so, we could see that the insurer equivalent utility premium calculation principle possesses the additivity property in the case of the linear insurer's utility function.

Let us now switch to the case of $U(x) = -\alpha e^{-\beta x} + \gamma,$ for $\min[\alpha, \beta] > 0.$ Here for any two independent risks X_1 and $X_2,$ and any insurer's initial capital $W,$ we get

$$-\alpha e^{-\beta W} + \gamma = \mathsf{E}[-\alpha e^{-\beta(W + \pi_{\text{i.e.u.}}[X_i] - X_i)} + \gamma], \quad \text{for} \quad i = \overline{1,2},$$

which yields

$$\pi_{\text{i.e.u.}}[X_i] = \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X_i}]) = \pi_{\exp(\beta)}[X_i], \quad \text{for} \quad i = \overline{1,2}.$$

Moreover

$$-\alpha e^{-\beta W} + \gamma = \mathsf{E}[-\alpha e^{-\beta(W + \pi_{\text{i.e.u.}}[X_1 + X_2] - X_1 - X_2)} + \gamma],$$

hence

$$\pi_{\text{i.e.u.}}[X_1 + X_2] = \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X_1}]) + \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X_2}]) = \pi_{\text{i.e.u.}}[X_1] + \pi_{\text{i.e.u.}}[X_2],$$

and as we have seen, the additivity property is possessed by the insurer equivalent utility premium calculation principle in the case of the exponential insurer's utility function.

Proof of the sufficiency was finished, so we can start to prove the necessity.

Observe that the insurer equivalent utility premium calculation principle is invariant with respect to the linear transformations of the function $U(x)$, i.e., the principle based on the utility function $U(x)$ and the principle based on the utility function $\overline{U}(x) = l_1 U(x) + l_2$, for $l_1 > 0$, will produce the same premiums. Here the condition $l_1 > 0$ is imposed because otherwise the assumption of positivity of first derivative of the function $\overline{U}(x)$ will vanish.

In order to simplify the computations, we will fix the value of the insurer's initial capital W, derive all possible representations (in the case when the insurer equivalent utility principle is additive) for the function $\overline{U}(x)$ with

$$l_1 = 1/U'(W) \quad \text{and} \quad l_2 = -U(W)/U'(W),$$

and then we will switch back to the function $U(x)$.

Observe that the just defined utility function $\overline{U}(x)$ satisfies the following boundary conditions

$$\overline{U}(W) = 0, \quad \overline{U}'(W) = 1, \quad \text{and} \quad \overline{U}''(W) = \kappa, \tag{7}$$

for some real constant $\kappa \le 0$.

Let us now consider a risk X which takes only two possible values, namely t (here t is any real number different from zero) and 0 with probabilities p and $1-p$ respectively. The risk X can be viewed as a random function of the parameters p and t, and, therefore, within the proof of Theorem 2.1 will be denoted as X_p^t.

Equation (1) based on the utility function $\overline{U}(x)$ for the risk X_p^t will have the following form

$$\overline{U}(W) = \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] - t)\cdot p + \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t])\cdot(1-p). \tag{8}$$

Putting $p = 1$ into (8), obtain

$$\overline{U}(W) = \overline{U}(W + \pi_{\text{i.e.u.}}[X_1^t] - t). \tag{9}$$

Since $\overline{U}'(x) > 0$ for all x, then from (9) it follows

$$\pi_{\text{i.e.u.}}[X_1^t] = t. \tag{10}$$

Let us calculate partial derivatives with respect to p from both sides of the equation (8)

$$\begin{aligned} 0 \;=\; & \overline{U}'(W + \pi_{\text{i.e.u.}}[X_p^t] - t)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot p + \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \\ & + \overline{U}'(W + \pi_{\text{i.e.u.}}[X_p^t])\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot(1-p) - \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t]). \end{aligned} \tag{11}$$

Substituting $p = 1$ into the equation (11) and using the identity (10), obtain

$$0 = \overline{U}'(W)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\bigg|_{p=1} + \overline{U}(W) - \overline{U}(W+t). \tag{12}$$

Putting boundary conditions $\overline{U}(W) = 0$ and $\overline{U}'(W) = 1$ into the equation (12), we obtain a representation for the partial derivative with respect to the parameter p of the premium at the point $p = 1$, namely,

$$\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\bigg|_{p=1} = \overline{U}(W+t). \tag{13}$$

Let us consider also a risk Y, independent of X, taking two possible values, namely h (here h is any non-zero real number) and 0 with probabilities q and $1-q$ respectively. Being a random function of the parameters h and q the risk Y within the proof of Theorem 2.1 will be denoted as Y_q^h. Using manipulations similar to those performed with the risk X_p^t, one can conclude that

$$\pi_{\text{i.e.u.}}[Y_1^h] = h, \tag{14}$$

and that the partial derivative with respect to the parameter q of the premium at the point $q = 1$ is

$$\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\bigg|_{q=1} = \overline{U}(W + h). \tag{15}$$

Now, let us look at a risk $Z_{p,q}^{t,h}$ defined in the following way

$$Z_{p,q}^{t,h} := X_p^t + Y_q^h.$$

The risk $Z_{p,q}^{t,h}$ will take the values $t + h$, t, h, and 0 with probabilities pq, $p(1-q)$, $(1-p)q$, and $(1-p)(1-q)$ respectively.

If the insurer equivalent utility principle is additive then the following must hold

$$\pi_{\text{i.e.u.}}[Z_{p,q}^{t,h}] = \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h].$$

In this case equation (1) for the risk $Z_{p,q}^{t,h}$ based on the function $\overline{U}(x)$ will have a form

$$\begin{aligned} \overline{U}(W) \ = \ & \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - t - h)\cdot p\cdot q \\ & + \ \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - t)\cdot p\cdot (1-q) \\ & + \ \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - h)\cdot (1-p)\cdot q \\ & + \ \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h])\cdot (1-p)\cdot (1-q). \end{aligned} \tag{16}$$

Let us now calculate partial derivatives with respect to p from both sides of the equation (16)

$$\begin{aligned} 0 \ = \ & \overline{U}'(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - t - h)\cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot p\cdot q \\ & + \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - t - h)\cdot q \\ & + \overline{U}'(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - t)\cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot p\cdot (1-q) \\ & + \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - t)\cdot (1-q) \\ & + \overline{U}'(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - h)\cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot (1-p)\cdot q \\ & - \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h] - h)\cdot q \\ & + \overline{U}'(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h])\cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot (1-p)\cdot (1-q) \\ & - \overline{U}(W + \pi_{\text{i.e.u.}}[X_p^t] + \pi_{\text{i.e.u.}}[Y_q^h])\cdot (1-q). \end{aligned} \tag{17}$$

The next step is to take partial derivatives with respect to the parameter q from both sides of the equation (17), here we get

$$
\begin{aligned}
0 \;=\; & \bar{U}''(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-t-h)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\cdot p\cdot q\\
&+\bar{U}'(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-t-h)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot p\\
&+\bar{U}'(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-t-h)\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\cdot q\\
&+\bar{U}(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-t-h)\\
&+\bar{U}''(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-t)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\cdot p\cdot(1-q)\\
&-\bar{U}'(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-t)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot p\\
&+\bar{U}'(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-t)\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\cdot(1-q)\\
&-\bar{U}(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-t)\\
&+\bar{U}''(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-h)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\cdot(1-p)\cdot q\\
&+\bar{U}'(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-h)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot(1-p)\\
&-\bar{U}'(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-h)\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\cdot q\\
&-\bar{U}(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]-h)\\
&+\bar{U}''(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h])\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\cdot(1-p)\cdot(1-q)\\
&-\bar{U}'(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h])\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\cdot(1-p)\\
&-\bar{U}'(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h])\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\cdot(1-q)\\
&+\bar{U}(W+\pi_{\text{i.e.u.}}[X_p^t]+\pi_{\text{i.e.u.}}[Y_q^h]).
\end{aligned}
$$

Putting $p=q=1$ into the just derived equation, and using the identities (10) and (14), obtain

$$
\begin{aligned}
0 \;=\; & \bar{U}''(W)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\Big|_{p=1}\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\Big|_{q=1}\\
&+\bar{U}'(W)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\Big|_{p=1}+\bar{U}'(W)\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\Big|_{q=1}\\
&-\bar{U}'(W+t)\cdot\frac{\partial}{\partial q}\pi_{\text{i.e.u.}}[Y_q^h]\Big|_{q=1}-\bar{U}'(W+h)\cdot\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\Big|_{p=1}\\
&+\bar{U}(W)-\bar{U}(W+t)-\bar{U}(W+h)+\bar{U}(W+t+h).
\end{aligned}
\tag{18}
$$

Substituting identities (13) and (15), as well as identities $\overline{U}(W)=0$, $\overline{U}'(W)=1$, and $\overline{U}''(W)=\kappa$, into the equation (18) we finally get an equation which the utility function $\overline{U}(x)$ must satisfy if the premium calculation principle is additive

$$0 = \overline{U}(W+h+t)+\kappa\cdot\overline{U}(W+t)\cdot\overline{U}(W+h)-\overline{U}'(W+t)\cdot\overline{U}(W+h)-\overline{U}'(W+h)\cdot\overline{U}(W+t). \qquad (19)$$

Solving the equation (19), we will investigate separately cases of $\kappa=0$ and $\kappa<0.$ We start from the case of $\kappa<0.$ Since $\overline{U}(\cdot)$ is a concave function, then the following inequality takes place

$$\frac{\overline{U}(W)+\overline{U}(W+2t)}{2}\le\overline{U}(W+t). \qquad (20)$$

Taking into account boundary condition $\overline{U}(W)=0,$ inequality (20) can be rewritten as

$$\overline{U}(W+2t)\le 2\overline{U}(W+t). \qquad (21)$$

Substituting $t=h$ into the equation (19), obtain

$$0=\overline{U}(W+2t)+\kappa\overline{U}^{2}(W+t)-2\overline{U}'(W+t)\cdot\overline{U}(W+t). \qquad (22)$$

In order to apply some asymptotic techniques, without of loss of generality, we assume for the moment that the value of the parameter t is strictly positive. Using inequality (21) and taking into account that $\overline{U}(W)=0$ as well as $\overline{U}'(x)>0,$ for $x\in\mathbb{R},$ from the equation (22) we get

$$\overline{U}'(W+t)=\frac{\overline{U}(W+2t)+\kappa\overline{U}^{2}(W+t)}{2\overline{U}(W+t)}\le\frac{2\overline{U}(W+t)+\kappa\overline{U}^{2}(W+t)}{2\overline{U}(W+t)}=1+\frac{\kappa}{2}\overline{U}(W+t). \qquad (23)$$

Since $\kappa<0,$ then from inequality (23) it follows that the function $\overline{U}(\cdot)$ must be bounded from above because otherwise the value $1+\kappa\overline{U}(W+t)/2$ would be negative for sufficiently large values of the parameter t and this would contradict with the assumption of positivity of first derivative of the function $\overline{U}(\cdot).$

Since the function $\overline{U}(\cdot)$ is increasing and bounded and $\overline{U}(W)=0$, then must exist a positive finite limit of $\overline{U}(W+t)$ as the parameter t tends to plus infinity and moreover, must exist the limit

$$\lim_{t\to+\infty}\overline{U}'(W+t)=0, \qquad (24)$$

in addition to that for all $h\in\mathbb{R}\setminus\{0\}$ there exist the limit

$$\lim_{t\to+\infty}\frac{\overline{U}(W+t+h)}{\overline{U}(W+t)}=\lim_{t\to+\infty}\overline{U}(W+t+h)\Big/\lim_{t\to+\infty}\overline{U}(W+t)=1. \qquad (25)$$

Dividing both parts of the equation (19) by $\overline{U}(W+t)$, switching to the limit when t tends to plus infinity, and using the limit relations (24) and (25), we obtain an equation

$$\overline{U}'(W+h)=\kappa\overline{U}(W+h)+1. \qquad (26)$$

The equation (26) can be rewritten in the following equivalent form

$$\frac{d[\kappa\overline{U}(W+h)+1]}{\kappa\overline{U}(W+h)+1}=\kappa dh. \qquad (27)$$

From the equation (27) it follows

$$\kappa\overline{U}(W+h)+1=e^{\kappa(W+h)}\cdot c,\quad \text{for some constant}\quad c\in\mathbb{R}. \qquad (28)$$

From (28), using boundary condition $\overline{U}(W)=0,$ (here, due to continuity of $\overline{U}(\cdot),$ the value $\overline{U}(W)$ can be defined as $\lim_{h\to0}\overline{U}(W+h)$) we get

$$\overline{U}(W+h)=\frac{e^{\kappa(W+h)}\cdot e^{-\kappa W}-1}{\kappa},\quad \text{for}\quad h\in\mathbb{R}\setminus\{0\}. \tag{29}$$

Due to continuity of the function $\overline{U}(\cdot)$, equation (29) can be rewritten in terms of $x\in\mathbb{R}$

$$\overline{U}(x)=\frac{e^{\kappa x}\cdot e^{-\kappa W}-1}{\kappa}. \tag{30}$$

Taking into account that $\overline{U}''(W)=\kappa$, using representation (30) and the transformation identity

$$\overline{U}(x)=l_1U(x)+l_2,\quad \text{for}\quad l_1=1/U'(W)\quad \text{and}\quad l_2=-U(W)/U'(W), \tag{31}$$

we finally get corresponding admissible representation for the original utility function $U(x)$

$$U(x)=\frac{U'(W)e^{-\overline{U}''(W)\cdot W}}{\overline{U}''(W)}\cdot e^{\overline{U}''(W)\cdot x}-\frac{U'(W)}{\overline{U}''(W)}+U(W). \tag{32}$$

From the representation (32) it follows that in the case of $\overline{U}''(W)<0$ the function $U(x)$ must be a function of the form

$$U(x)=-\alpha e^{-\beta x}+\gamma$$

for some real constants α, β, and γ. Moreover conditions $U'(W)>0$ and $\overline{U}''(W)<0$ imply additional restrictions on the parameters α and β, namely, both of them must be strictly positive constants, or equivalently, $\min[\alpha,\beta]>0$.

Let us now switch to the case of $\kappa=0$. In this case the equation (19) will be simplified to the following one

$$0=\overline{U}(W+t+h)-\overline{U}'(W+t)\cdot\overline{U}(W+h)-\overline{U}'(W+h)\cdot\overline{U}(W+t). \tag{33}$$

Let us assume for a moment that the function $\overline{U}(\cdot)$ is bounded from above. Since derivative of the function $\overline{U}(\cdot)$ is strictly positive, then in the assumed case must exist the limit

$$\lim_{t\to+\infty}\overline{U}'(W+t)=0. \tag{34}$$

If the function $\overline{U}(\cdot)$ is increasing and bounded with $\overline{U}(W)=0$ then there exist a finite limit

$$\lim_{t\to+\infty}\overline{U}(W+t)=c>0, \tag{35}$$

and, moreover, for all $h\in\mathbb{R}\setminus\{0\}$ there exist the limit

$$\lim_{t\to+\infty}\frac{\overline{U}(W+t+h)}{\overline{U}(W+t)}=\lim_{t\to+\infty}\overline{U}(W+t+h)\Big/\lim_{t\to+\infty}\overline{U}(W+t)=1. \tag{36}$$

Combination of (34) and (35) yields

$$\lim_{t\to+\infty}\frac{\overline{U}'(W+t)}{\overline{U}(W+t)}=\lim_{t\to+\infty}\overline{U}'(W+t)\Big/\lim_{t\to+\infty}\overline{U}(W+t)=0. \tag{37}$$

Dividing both sides of the equation (33) by $\overline{U}(W+t)$, switching to the limit as the parameter t tends to plus infinity, and using the limit relations (36) and (37), obtain

$$\overline{U}'(W+h)=1,\quad \text{for all } h\in\mathbb{R}\setminus\{0\}. \tag{38}$$

The equation (38) contradicts with the limit relation (34). This means that the assumption of boundedness from above of the function $\overline{U}(\cdot)$ in the case of $\kappa=0$ was wrong. Therefore in the case of $\kappa=0$ must hold the following limit relation

$$\lim_{t\to+\infty}\overline{U}(W+t)=+\infty. \tag{39}$$

Since we analyze the limit behavior of the equation (33) when the parameter t tends to plus infinity, then for any fixed $h \in \mathbb{R} \setminus \{0\}$, without of loss of generality, we may assume that the parameter t is larger than the parameter h. Due to concavity of the function $\overline{U}(\cdot)$, for any $h > 0$, from inequality

$$W + h < W + t < W + t + h$$

for any $\theta \in [0, 1]$ it follows

$$\overline{U}(\theta(W+h) + (1-\theta)(W+t+h)) \ \geq \ \theta\overline{U}(W+h) + (1-\theta)\overline{U}(W+t+h). \tag{40}$$

Putting $\theta = h/t$ into (40), obtain

$$\overline{U}\left(\frac{h}{t}(W+h) + \frac{(t-h)(W+t+h)}{t}\right) \ \geq \ \frac{h}{t}\overline{U}(W+h) + \frac{t-h}{t}\overline{U}(W+t+h)$$

or equivalently

$$\overline{U}(W+t) \ \geq \ \frac{h}{t}\overline{U}(W+h) + \frac{t-h}{h}\overline{U}(W+t+h). \tag{41}$$

It is more convenient for us to write inequality (41) in the following form

$$\frac{t}{t-h}\overline{U}(W+t) - \frac{h}{t-h}\overline{U}(W+h) \ \geq \ \overline{U}(W+t+h). \tag{42}$$

Observe that

$$\frac{t}{t-h}\overline{U}(W+t) - \frac{h}{t-h}\overline{U}(W+h) \ \sim \ \overline{U}(W+t) \quad \text{as} \quad t \to +\infty. \tag{43}$$

Here notation

$$g_1(t) \sim g_2(t) \quad \text{as} \quad t \to +\infty$$

means

$$\lim_{t\to+\infty} \frac{g_1(t)}{g_2(t)} = 1.$$

Since derivative of the function $\overline{U}(\cdot)$ is strictly positive, then, for any $h > 0$ and any admissible t, holds inequality

$$\overline{U}(W+t+h) > \overline{U}(W+t). \tag{44}$$

Combination of (42) and (44) yields

$$\frac{t}{t-h}\overline{U}(W+t) - \frac{h}{t-h}\overline{U}(W+h) \ \geq \ \overline{U}(W+t+h) \ > \ \overline{U}(W+t). \tag{45}$$

Switching to the limit as $t \to +\infty$ and using (43), obtain

$$\lim_{t\to+\infty} \frac{\overline{U}(W+t+h)}{\overline{U}(W+t)} = 1, \quad \text{for all } h > 0. \tag{46}$$

In the case of $h < 0$ for t large enough (namely for $t > 0$) hold inequalities

$$W + h < W + t + h < W + t. \tag{47}$$

From the inequalities (47) due to concavity of the function $\overline{U}(\cdot)$ for any $\theta \in [0, 1]$ it follows

$$\overline{U}(\theta(W+h) + (1-\theta)(W+t)) \ \geq \ \theta\overline{U}(W+h) + (1-\theta)\overline{U}(W+t). \tag{48}$$

Substituting $\theta = h/(h-t)$ into the inequality (48), obtain

$$\overline{U}(W+t+h) \ \geq \ \frac{h}{h-t}\overline{U}(W+h) + \frac{t}{t-h}\overline{U}(W+t). \tag{49}$$

Observe that

$$\frac{h}{h-t}\overline{U}(W+h) + \frac{t}{t-h}\overline{U}(W+t) \ \sim \ \overline{U}(W+t) \quad \text{as} \quad t \to +\infty. \tag{50}$$

Since $h<0$ and $\overline{U}(\cdot)$ is an increasing function, then

$$\overline{U}(W+t)>\overline{U}(W+t+h). \tag{51}$$

Combination of (49) and (51) yields

$$\overline{U}(W+t) \;>\; \overline{U}(W+t+h) \;\geq\; \frac{h}{h-t}\overline{U}(W+h)+\frac{t}{t-h}\overline{U}(W+t). \tag{52}$$

Switching to the limit as t tends to plus infinity and using the limit relation (50), we get

$$\lim_{t\to+\infty}\frac{\overline{U}(W+t+h)}{\overline{U}(W+t)}=1, \quad \text{for all } h<0. \tag{53}$$

Since $\overline{U}'(\cdot)$ is a non-increasing (because $\overline{U}''(\cdot)\leq 0$) positive function, then from the limit relation (39) it follows

$$\lim_{t\to+\infty}\frac{\overline{U}'(W+t)}{\overline{U}(W+t)}=0. \tag{54}$$

Dividing both sides of the equation (33) by $\overline{U}(W+t)$, switching to the limit as t tends to plus infinity and using the limit relations (46), (53), and (54), obtain

$$\overline{U}'(W+h)=1, \quad \text{for} \quad h\in\mathbb{R}\setminus\{0\}. \tag{55}$$

Due to continuity of the function $\overline{U}'(\cdot)$, equation (55) can be rewritten in terms of the parameter $x\in\mathbb{R}$, namely,

$$\overline{U}'(x)=1. \tag{56}$$

From the equation (56), using boundary condition $\overline{U}(W)=0$, we obtain the second admissible representation for the utility function $\overline{U}(\cdot)$

$$\overline{U}(x)=x-W. \tag{57}$$

Combining (57) with the transformation identity

$$\overline{U}(x)=l_1U(x)+l_2, \quad \text{for} \quad l_1=1/U'(W) \quad \text{and} \quad l_2=-U(W)/U'(W), \tag{58}$$

we finally get corresponding admissible representation for the original utility function $U(x)$

$$U(x)=U'(W)(x-W)+u(W). \tag{59}$$

Equation (59) means that the tangent straight line to the function $U(x)$ at the point W coincides with the function $U(x)$ itself, and hence the function $U(x)$ must be a function of the form

$$U(x)=ax+b$$

for some real constants a and b. Moreover, the constant a must be a strictly positive constant because otherwise this would contradict with the assumption of positivity of first derivative of the function $U(x)$.

This completes the proof of Theorem 2.1. □

Let us now show that the insurer equivalent utility premium principle coincides with the net premium principle if and only if $U(x)=ax+b$, for $a>0$, and coincides with the exponential premium principle if and only if $U(x)=-\alpha e^{-\beta x}+\gamma$, for $\min[\alpha,\beta]>0$.

For this reason we will need the following inequality

$$\pi_{\exp(\beta)}[X]=\frac{1}{\beta}\log(\mathsf{E}[e^{\beta X}]) \;\geq\; \frac{1}{\beta}\log(e^{\beta\mathsf{E}[X]})=\mathsf{E}[X]=\pi_{\text{net}}[X],$$

and, moreover, exact equality in the inequality $\mathsf{E}[e^{\beta X}]\geq e^{\beta\mathsf{E}[X]}$ appears if and only if $\mathrm{P}\{X=C\}=1$ for some constant $C\in\mathbb{R}$. Therefore, generally speaking, the net premium principle is not a special case of the exponential premium principle and viceversa.

Let us now assume that for some function $U(x)$, different from the exponential function, the insurer equivalent utility premium principle will be equivalent to the exponential premium principle. Then, due to additivity of the exponential premium principle, such method of pricing must be additive. However in the proof of Theorem 2.1 was shown that the insurer equivalent utility premium calculation principle is additive if and only if $U(x) = ax + b$, for $a > 0$, and $U(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$. Here the case of $U(x) = ax + b$, for $a > 0$, corresponds to the net premium principle, which, as was demonstrated, generally speaking is not a special case of the exponential premium principle. As we see, original assumption about the existence of a non-exponential insurer's utility function $U(x)$ which would produce a principle equivalent to the exponential premium principle leads to a contradiction. Therefore, the case of $U(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, is indeed the only case when the insurer equivalent utility premium principle is equivalent to the exponential premium principle.

Using similar contradiction technique one can conclude that the case of $U(x) = ax + b$, for $a > 0$, is the only case when the insurer equivalent utility premium principle is equivalent to the net premium principle.

Since the insurer's initial capital in the proof of Theorem 2.1 was chosen arbitrary and no restriction on it had been used within the proof, then we can formulate the following corollary to Theorem 2.1.

Corollary 2.1. The *insurer zero utility premium calculation principle possesses the additivity property if and only if* $U(x) = ax + b$, *for* $a > 0$, *or* $U(x) = -\alpha e^{-\beta x} + \gamma$, *for* $\min[\alpha, \beta] > 0$, *i.e., only in the cases when it coincides with either the net premium principle or the exponential premium principle.*

3. Iterativity Property

The following theorem describes the necessary and sufficient conditions imposed on the insurer's utility function under which the iterativity property is possessed by the insurer equivalent utility premium calculation principle.

Theorem 3.1. *The insurer equivalent utility premium calculation principle possesses the iterativity property if and only if* $U(x) = ax + b$, *for* $a > 0$, *or* $U(x) = -\alpha e^{-\beta x} + \gamma$, *for* $\min[\alpha, \beta] > 0$, *i.e., only in the cases when it coincides with either the net premium principle or the exponential premium principle.*

Proof. Let us from the beginning prove the sufficiency of the statement. We start from the linear utility function $U(x) = ax + b$, for $a > 0$, and show that in this case the insurer equivalent utility premium principle is equivalent to the net principle. Indeed, here for any risk X, and any insurer's initial capital W, equation (1) will be simplified to the following one

$$aW + b = \mathsf{E}[a(W + \pi_{\text{i.e.u.}}[X] - X) + b],$$

therefore, in the case of the linear insurer's utility function,

$$\pi_{\text{i.e.u.}}[X] = \mathsf{E}[X] = \pi_{\text{net}}[X].$$

Then, for any two risks X and Y, any insure's initial capital W, and the same insurer's utility function, form the equation (1) we get

$$aW + b = aW + a\pi_{\text{i.e.u.}}[X \mid Y] - a\mathsf{E}[X \mid Y] + b,$$

therefore, in the case of the linear insurer's utility function,

$$\pi_{\text{i.e.u.}}[X \mid Y] = \mathsf{E}[X \mid Y],$$

moreover, from the same equation, it follows

$$aW + b = aW + a\pi_{\text{i.e.u.}}[\pi_{\text{i.e.u.}}[X \mid Y]] - a\mathsf{E}[\pi_{\text{i.e.u.}}[X \mid Y]] + b,$$

and we finally get

$$\pi_{\text{i.e.u.}}[\pi_{\text{i.e.u.}}[X \mid Y]] = \mathsf{E}[\pi_{\text{i.e.u.}}[X \mid Y]] = \mathsf{E}[\mathsf{E}[X \mid Y]] = \mathsf{E}[X] = \pi_{\text{net}}[X] = \pi_{\text{i.e.u.}}[X]$$

thus, the iterativity property is possessed by the insurer equivalent utility principle in the case of the linear insure's utility function.

Let us now switch to the case of the exponential insurer's utility function, i.e., the case of $U(x) = -\alpha e^{-\beta x} + \gamma,$ for $\min[\alpha, \beta] > 0.$ We show first that this is the case when the insurer equivalent utility premium principle is equivalent to the exponential premium principle. Here for any risk X and any insurer's initial capital W from the equation (1) we get

$$-\alpha e^{-\beta W} + \gamma = \mathsf{E}[-\alpha e^{-\beta(W + \pi_{\text{i.e.u.}}[X] - X)} + \gamma],$$

which yields

$$\pi_{\text{i.e.u.}}[X] = \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X}]) = \pi_{\exp(\beta)}[X].$$

Then, for any two risks X and $Y,$ any insurer's initial capital $W,$ and the same exponential utility function, from the equation (1) we get

$$-\alpha e^{-\beta W} + \gamma = -\alpha e^{-\beta W} \cdot e^{-\beta\pi_{\text{i.e.u.}}[X|Y]} \cdot \mathsf{E}[e^{-\beta X} \mid Y] + \gamma,$$

therefore, in the case of the exponential insurer's utility function

$$\pi_{\text{i.e.u.}}[X] = \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X}]) = \pi_{\exp(\beta)}[X].$$

Moreover, from the equation (1) for the risk $\pi_{\text{i.e.u.}}[X \mid Y]$ in the case of the exponential utility function, namely,

$$-\alpha e^{-\beta W} + \gamma = -\alpha e^{-\beta W} \cdot e^{-\beta\pi_{\text{i.e.u.}}[\pi_{\text{i.e.u.}}[X|Y]]} \cdot \mathsf{E}[e^{-\beta\pi_{\text{i.e.u.}}[X|Y]}] + \gamma,$$

obtain

$$\begin{aligned}\pi_{\text{i.e.u.}}[\pi_{\text{i.e.u.}}[X \mid Y]] &= \frac{1}{\beta}\log(\mathsf{E}[e^{-\beta\pi_{\text{i.e.u.}}[X|Y]}]) = \frac{1}{\beta}\log(\mathsf{E}[e^{-\beta\cdot\frac{1}{\beta}\log(\mathsf{E}[e^{-\beta X}|Y])}]) \\ &= \frac{1}{\beta}\log(\mathsf{E}[\mathsf{E}[e^{-\beta X} \mid Y]]) = \frac{1}{\beta}\log(\mathsf{E}[e^{-\beta X}]) = \pi_{\exp(\beta)}[X] \\ &= \pi_{\text{i.e.u.}}[X],\end{aligned}$$

hence, the iterativity property is also possessed by the insurer equivalent utility principle in the case of the exponential utility function.

Proof of the sufficiency was finished, so we can start to prove the necessity.

Let us now consider a risk X taking only two possible values, namely t (here t is any real number different from zero) and 0 with probabilities p and $1-p$ respectively. Being a random function of the parameters t and p within the proof of Theorem 3.1 the risk X will be denoted as X_p^t.

Equivalent utility equation (1) for any insurer's initial capital W and any insurer's utility function $U(\cdot)$ for the risk X_p^t will have a form

$$U(W) = U(W + \pi_{\text{i.e.u.}}[X_p^t] - t)\cdot p + U(W + \pi_{\text{i.e.u.}}[X_p^t])\cdot(1-p). \tag{60}$$

From the equation (60), assigning $p = 0$, we get

$$U(W) = U(W + \pi_{\text{i.e.u.}}[X_0^t]). \tag{61}$$

Since $U'(x) > 0$ for all $x,$ then from the equation (61) it follows

$$\pi_{\text{i.e.u.}}[X_0^t] = 0. \tag{62}$$

Substituting $p=1$ into the equation (60), obtain

$$U(W) = U(W + \pi_{\text{i.e.u.}}[X_1^t] - t), \tag{63}$$

again, since $U'(x) > 0$ for all x, then from the equation (63) we get

$$\pi_{\text{i.e.u.}}[X_1^t] = t. \tag{64}$$

Differentiation of the equation (60) with respect to the parameter p yields

$$\begin{aligned} 0 \;=\; & U'(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t] \cdot p + U(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \\ & + U'(W + \pi_{\text{i.e.u.}}[X_p^t]) \cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t] \cdot (1-p) - U(W + \pi_{\text{i.e.u.}}[X_p^t]). \end{aligned} \tag{65}$$

Taking into account strict monotonicity of the function $U(\cdot)$, the equation (65) can be rewritten in the following way

$$\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t] = \frac{U(W + \pi_{\text{i.e.u.}}[X_p^t]) - U(W + \pi_{\text{i.e.u.}}[X_p^t] - t)}{U'(W + \pi_{\text{i.e.u.}}[X_p^t] - t)p + U'(W + \pi_{\text{i.e.u.}}[X_p^t])(1-p)}. \tag{66}$$

From the representation (66) it follows

$$\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t] \neq 0, \quad \text{for} \quad t \neq 0. \tag{67}$$

Substituting $p=0$ into the equation (65), using the identity (62), we get an equation which will be used a bit later, namely,

$$0 = U(W-t) - U(W) + U'(W) \cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\bigg|_{p=0}. \tag{68}$$

Let Y be a variable taking values h_1 and h_2 (here h_i, for $i = \overline{1,2}$, are arbitrary real numbers from the interval [0, 1]) with probabilities $1/2$ and $1/2$. And let $\overline{X}$ be a risk with the following conditional distribution, defined under known values of the variable Y,

$$\mathrm{P}\{\overline{X} = t | Y = h_i\} = h_i, \quad \text{for} \quad i = \overline{1,2};$$

$$\mathrm{P}\{\overline{X} = 0 | Y = h_i\} = 1 - h_i, \quad \text{for} \quad i = \overline{1,2}.$$

Using the formula of complete probability, we can find out the unconditional distribution of the risk $\overline{X}$, namely,

$$\mathrm{P}\{\overline{X} = t\} = \sum_{i=1}^{2} \mathrm{P}\{\overline{X} = t | Y = h_i\} \cdot \mathrm{P}\{Y = h_i\} = \frac{h_1 + h_2}{2};$$

$$\mathrm{P}\{\overline{X} = 0\} = \sum_{i=1}^{2} \mathrm{P}\{\overline{X} = 0 | Y = h_i\} \cdot \mathrm{P}\{Y = h_i\} = 1 - \frac{h_1 + h_2}{2}.$$

As we see, unconditional distribution of the risk $\overline{X}$ coincides with distribution of the risk $X_{\overline{p}}^t$, where $\overline{p} = (h_1 + h_2)/2$, therefore, the following identity must hold

$$\pi_{\text{i.e.u.}}[\overline{X}] = \pi_{\text{i.e.u.}}[X_{\overline{p}}^t]. \tag{69}$$

Observe that the equivalent utility equation (1) for $\pi_{\text{i.e.u.}}[\overline{X}|Y = h_i]$, here $i = \overline{1,2}$, namely,

$$U(W) = U(W + \pi_{\text{i.e.u.}}[\overline{X}|Y = h_i] - t) \cdot h_i + U(W + \pi_{\text{i.e.u.}}[\overline{X}|Y = h_i]) \cdot (1 - h_i),$$

coincides with the equivalent utility equation for $\pi_{\text{i.e.u.}}[X_{h_i}^t]$, here also $i = \overline{1,2}$, namely,

$$U(W) = U(W + \pi_{\text{i.e.u.}}[X_{h_i}^t] - t)\cdot h_i + U(W + \pi_{\text{i.e.u.}}[X_{h_i}^t])\cdot(1-h_i),$$

therefore, the following identity must hold

$$\pi_{\text{i.e.u.}}[\overline{X}|Y = h_i] = \pi_{\text{i.e.u.}}[X_{h_i}^t], \quad \text{for} \quad i = \overline{1,2}, \tag{70}$$

moreover, in the case of the iterative insurer equivalent utility premium principle

$$\pi_{\text{i.e.u.}}[\overline{X}] = \pi_{\text{i.e.u.}}[\pi_{\text{i.e.u.}}[\overline{X}\,|\,Y]]. \tag{71}$$

The identity (71), in combination with (1), implies the equivalent utility equation

$$U(W) = \mathrm{E}[U(W + \pi_{\text{i.e.u.}}[\overline{X}] - \pi_{\text{i.e.u.}}[\overline{X}\,|\,Y])]. \tag{72}$$

Taking into account possible values of the variable Y, the equation (72) can be rewritten in the following equivalent form

$$U(W) = \frac{1}{2}\sum_{i=1}^{2} U(W + \pi_{\text{i.e.u.}}[\overline{X}] - \pi_{\text{i.e.u.}}[\overline{X}\,|\,Y = h_i]).$$

Taking into account identities (69) and (70) we can modify the equation (72) a bit further

$$U(W) = \frac{1}{2}\sum_{i=1}^{2} U(W + \pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \pi_{\text{i.e.u.}}[X_{h_i}^t]). \tag{73}$$

Let us now differentiate equation (73) with respect to h_1, obtain

$$\begin{aligned} 0 \;=\; & \frac{1}{2}U'(W + \pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \pi_{\text{i.e.u.}}[X_{h_1}^t])\cdot\left[\frac{1}{2}\frac{\partial}{\partial h_1}\pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \frac{\partial}{\partial h_1}\pi_{\text{i.e.u.}}[X_{h_1}^t]\right] \\ & + \frac{1}{2}U'(W + \pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \pi_{\text{i.e.u.}}[X_{h_2}^t])\cdot\frac{1}{2}\frac{\partial}{\partial h_1}\pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t]. \end{aligned}$$

Let us also differentiate just obtained equation with respect to h_1, here we get

$$\begin{aligned} 0 \;=\; & \frac{1}{2}U''(W + \pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \pi_{\text{i.e.u.}}[X_{h_1}^t])\cdot\left[\frac{1}{2}\frac{\partial}{\partial h_1}\pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \frac{\partial}{\partial h_1}\pi_{\text{i.e.u.}}[X_{h_1}^t]\right]^2 \\ & + \frac{1}{2}U'(W + \pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \pi_{\text{i.e.u.}}[X_{h_1}^t])\cdot\frac{1}{4}\frac{\partial^2}{(\partial h_1)^2}\pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] \\ & - \frac{1}{2}U'(W + \pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \pi_{\text{i.e.u.}}[X_{h_1}^t])\cdot\frac{\partial^2}{(\partial h_1)^2}\pi_{\text{i.e.u.}}[X_{h_1}^t] \\ & + \frac{1}{2}U''(W + \pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \pi_{\text{i.e.u.}}[X_{h_2}^t])\cdot\left[\frac{1}{2}\frac{\partial}{\partial h_1}\pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t]\right]^2 \\ & + \frac{1}{2}U'(W + \pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t] - \pi_{\text{i.e.u.}}[X_{h_2}^t])\cdot\frac{1}{4}\frac{\partial^2}{(\partial h_1)^2}\pi_{\text{i.e.u.}}[X_{\frac{h_1+h_2}{2}}^t]. \end{aligned}$$

Putting $h_1 = h_2 =: h$ into the last equation, obtain

$$\begin{aligned} 0 \;=\; & \frac{1}{2}U''(W)\cdot\left[\frac{1}{2}\frac{\partial}{\partial h}\pi_{\text{i.e.u.}}[X_h^t]\right]^2 + \frac{1}{2}U'(W)\cdot\left[-\frac{3}{4}\frac{\partial^2}{(\partial h)^2}\pi_{\text{i.e.u.}}[X_h^t]\right] \\ & + \frac{1}{2}U''(W)\cdot\left[\frac{1}{2}\frac{\partial}{\partial h}\pi_{\text{i.e.u.}}[X_h^t]\right] + \frac{1}{2}U'(W)\cdot\left[\frac{1}{4}\frac{\partial^2}{(\partial h)^2}\pi_{\text{i.e.u.}}[X_h^t]\right], \end{aligned} \tag{74}$$

simplifying (74) we get differential equation for $\pi_{\text{i.e.u.}}[X_h^t]$ as a function of the parameter h defined for $0 \le h \le 1$, namely,

$$U''(W) \cdot \left[\frac{\partial}{\partial h} \pi_{\text{i.e.u.}}[X_h^t] \right]^2 = U'(W) \cdot \frac{\partial^2}{(\partial h)^2} \pi_{\text{i.e.u.}}[X_h^t]. \tag{75}$$

with boundary conditions, which follow from (62) and (64),

$$\pi_{\text{i.e.u.}}[X_0^t] = 0 \quad \text{and} \quad \pi_{\text{i.e.u.}}[X_1^t] = t. \tag{76}$$

Since the function $U(\cdot)$ is a concave function, then $U''(W) \le 0$. Let us solve the equation (75) separately for the cases of $U''(W) < 0$ and $U''(W) = 0$. We start from the case of $U''(W) = 0$. In this case the equation (74) will be simplified to the following one

$$\frac{\partial^2}{(\partial h)^2} \pi_{\text{i.e.u.}}[X_h^t] = 0. \tag{77}$$

Solution to the equation (77) must have a form

$$\pi_{\text{i.e.u.}}[X_h^t] = \kappa_1 h + \kappa_2, \quad \text{for some constants } \kappa_1 \text{ and } \kappa_2. \tag{78}$$

Applying boundary conditions (76) to the solution (78) we see that the solution to the equation (75) with boundary conditions (76) in the case of $U''(W) = 0$ is

$$\pi_{\text{i.e.u.}}[X_h^t] = ht. \tag{79}$$

Differentiatiation of the obtained solution (79) with respect to h at the point $h = 0$ yields

$$\left. \frac{\partial}{\partial h} \pi_{\text{i.e.u.}}[X_h^t] \right|_{h=0} = t. \tag{80}$$

Substituting representation (80) into the equation (68), we finally get an equation which the utility function $U(\cdot)$ has to satisfy in the case of the iterative insurer equivalent utility premium calculation principle, namely,

$$0 = U(W - t) - U(W) + U'(W)t. \tag{81}$$

The variable t was taken from $\mathbb{R} \setminus \{0\}$, however, due to continuity of the function $U(\cdot)$, making substitution $W - t =: x$, equation (81) can be rewritten in terms of the original parameter $x \in \mathbb{R}$:

$$U(x) = U'(W) \cdot (x - W) + U(W). \tag{82}$$

Representation (82) can be interpreted as follows: the tangent straight line to the function $U(\cdot)$ at the point $x = W$ coincides with the function $U(\cdot)$ itself, therefore, in the case of $U''(W) = 0$, the function $U(\cdot)$ must be a function of the form

$$U(x) = ax + b,$$

for some real constants a and b. Assumption of $U'(W) > 0$, which follows from the original assumption of positivity of first derivative of the function $U(\cdot)$, gives us additional restriction on the parameter a: parameter a must be a strictly positive constant.

Let us now solve the equation (75) in the case of $U''(W) < 0$. For the computational convenience we make a substitution

$$Z(h) := \frac{\partial}{\partial h} \pi_{\text{i.e.u.}}[X_h^t],$$

and rewrite the equation (75) in the following form

$$U''(W) \cdot Z^2(h) = U'(W) Z'(h). \tag{83}$$

Taking into account $U'(W) > 0$ as well as representation (67), with replacement of the parameter p by the parameter h, equation (83) can be rewritten in the following way

$$\frac{dZ}{Z^2}=\frac{U''(W)}{U'(W)}dh. \tag{84}$$

Solution to the equation (84) is

$$-Z^{-1}(h)=\frac{U''(W)}{U'(W)}h-\kappa_1, \quad \text{for some constant } \kappa_1.$$

Switching back to $\pi_{\text{i.e.u.}}[X_h^t]$, obtain

$$\frac{\partial}{\partial h}\pi_{\text{i.e.u.}}[X_h^t]=\frac{1}{\kappa_1-\frac{U''(W)}{U'(W)}h}. \tag{85}$$

Equation (85) can be slightly modified to the following one

$$d\pi_{\text{i.e.u.}}[X_h^t]=-\frac{U'(W)}{U''(W)}\cdot\frac{d\left[\kappa_1-\frac{U''(W)}{U'(W)}h\right]}{\kappa_1-\frac{U''(W)}{U'(W)}h}. \tag{86}$$

Solution to the equation (86) is

$$\pi_{\text{i.e.u.}}[X_h^t]=-\frac{U'(W)}{U''(W)}\cdot\log\left|\kappa_1-\frac{U''(W)}{U'(W)}h\right|+\kappa_2, \quad \text{for some constant } \kappa_2,$$

or equivalently,

$$\pi_{\text{i.e.u.}}[X_h^t]=-\frac{U'(W)}{U''(W)}\cdot\log\left|\left(\kappa_1-\frac{U''(W)}{U'(W)}h\right)e^{-\kappa_2\frac{U''(W)}{U'(W)}}\right|. \tag{87}$$

Applying boundary condition $\pi_{\text{i.e.u.}}[X_0^t]=0$ to the solution (87), we get

$$\log\left|\left(\kappa_1-\frac{U''(W)}{U'(W)}\cdot 0\right)e^{-\kappa_2\frac{U''(W)}{U'(W)}}\right|=0,$$

and hence

$$\kappa_1\cdot e^{-\kappa_2\frac{U''(W)}{U'(W)}}=1, \quad \text{which means that} \quad \kappa_1=e^{\kappa_2\frac{U''(W)}{U'(W)}}.$$

Using just obtained values of κ_1, solution (87) can be rewritten in the following way

$$\pi_{\text{i.e.u.}}[X_h^t]=-\frac{U'(W)}{U''(W)}\cdot\log\left|1-\frac{U''(W)}{U'(W)}\cdot h\cdot e^{-\kappa_2\frac{U''(W)}{U'(W)}}\right|. \tag{88}$$

Application of the boundary condition $\pi_{\text{i.e.u.}}[X_1^t]=t$ to the solution (88) implies

$$\log\left|1-\frac{U''(W)}{U'(W)}\cdot 1\cdot e^{-\kappa_2\frac{U''(W)}{U'(W)}}\right|=-t\cdot\frac{U''(W)}{U'(W)}. \tag{89}$$

Since $U'(W)>0$ and at the moment we consider the case of $U''(W)<0$, then

$$\frac{U''(W)}{U'(W)}<0, \quad \text{hence,} \quad 1-\frac{U''(W)}{U'(W)}\cdot e^{-\kappa_2\frac{U''(W)}{U'(W)}}>0,$$

therefore, identity (89) can be rewritten in the following way

$$-\frac{U''(W)}{U'(W)}\cdot e^{-\kappa_2 \frac{U''(W)}{U'(W)}} = e^{-t\cdot\frac{U''(W)}{U'(W)}} - 1. \tag{90}$$

Combining solution (88) with the identity (90), we finally get the solution to the equation (75) satisfying boundary conditions (76) in the case of $U''(W) < 0$, namely,

$$\pi_{\text{i.e.u.}}[X_h^t] = -\frac{U'(W)}{U''(W)}\log\left|1 + h\left(e^{-t\cdot\frac{U''(W)}{U'(W)}} - 1\right)\right|$$

Knowing $\pi_{\text{i.e.u.}}[X_h^t]$, we get

$$\frac{\partial}{\partial h}\pi_{\text{i.e.u.}}[X_h^t] = -\frac{U'(W)}{U''(W)}\cdot\frac{e^{-t\cdot\frac{U''(W)}{U'(W)}} - 1}{1 - h + he^{-t\cdot\frac{U''(W)}{U'(W)}}}. \tag{91}$$

From (91) it follows

$$\left.\frac{\partial}{\partial h}\pi_{\text{i.e.u.}}[X_h^t]\right|_{h=0} = -\frac{U'(W)}{U''(W)}\cdot\left(e^{-t\cdot\frac{U''(W)}{U'(W)}} - 1\right). \tag{92}$$

Combining (92) with the equation (68), we finally get an equation which the function $U(\cdot)$ has to satisfy in the case of the iterative insurer equivalent utility premium calculation principle, namely,

$$0 = U(W-t) - U(W) - U'(W)\cdot\frac{U'(W)}{U''(W)}\cdot\left(e^{-t\cdot\frac{U''(W)}{U'(W)}} - 1\right). \tag{93}$$

The parameter t was taken from $\mathbb{R}\setminus\{0\}$, however, due to continuity of the function $U(\cdot)$, making substitution $W - t =: x$, equation (93) can be rewritten in terms of the original parameter $x \in \mathbb{R}$:

$$U(x) = \frac{(U'(W))^2}{U''(W)}\cdot e^{-W\frac{U''(W)}{U'(W)}}\cdot e^{x\frac{U''(W)}{U'(W)}} - \frac{(U'(W))^2}{U''(W)} + U(W). \tag{94}$$

From the representation (94) it follows that in the case of $U''(W) < 0$ the function $U(x)$ must be a function of the form

$$U(x) = -\alpha e^{-\beta x} + \gamma$$

for some real constants α, β, and γ. Moreover conditions $U'(W) > 0$ and $U''(W) < 0$ imply additional restrictions on the parameters α and β, namely, both of them must be strictly positive constants, or equivalently, $\min[\alpha,\beta] > 0$.

This completes the proof of Theorem 3.1. □

In a way similar to the one presented in the previous section, the proof of Theorem 3.1 can be used for showing that the case of $U(x) = ax + b$, for $a > 0$, is the only case when the insurer equivalent utility premium principle is equivalent to the net premium principle and that $U(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha,\beta] > 0$, is the only case when the insurer equivalent utility premium principle is equivalent to the exponential premium principle.

Since we did not use any restrictions on the insurer's initial capital within the proof of Theorem 3.1, then we can formulate the following corollary to Theorem 3.1.

Corollary 3.1. *The insurer zero utility premium calculation principle possesses the iterativity property if and only if* $U(x) = ax + b$, *for* $a > 0$, *or* $U(x) = -\alpha e^{-\beta x} + \gamma$, *for* $\min[\alpha, \beta] > 0$, *i.e., only in the cases when it coincides with either the net premium principle or the exponential premium principle.*

4. Consistency Property

In contrast to the customer equivalent utility premium calculation principle which possesses the consistency property only for some special choices of the utility function, the insurer equivalent utility principle (and also, as a consequence, the insurer zero utility principle) possesses the consistency property with arbitrary choice of the utility function. Indeed, since for any risk X, any insurer's initial capital W, any insurer's utility function $U(\cdot)$, and any $c \in \mathbb{R}$ hold identities

$$\begin{aligned} U(W) &= \mathsf{E}[U(W + \pi_{\text{i.e.u.}}[X + c] - (c + X))] \\ &= \mathsf{E}[U(W + (\pi_{\text{i.e.u.}}[X + c] - c) - X)] \\ &= \mathsf{E}[U(W + \pi_{\text{i.e.u.}}[X] - X)] = U(W), \end{aligned}$$

then

$$\pi_{\text{i.e.u.}}[X + c] - c = \pi_{\text{i.e.u.}}[X], \quad \text{or equivalently,} \quad \pi_{\text{i.e.u.}}[X + c] = \pi_{\text{i.e.u.}}[X] + c,$$

so, we observe the fulfillment of the mentioned property.

5. Scale Invariance Property

The following theorem describes the necessary and sufficient conditions of attainment of the scale invariance property by the insurer equivalent utility premium calculation principle.

Theorem 5.1. *The insurer equivalent utility premium calculation principle possesses the scale invariance property if and only if* $U(x) = ax + b$, *for* $a > 0$, *i.e., only in the case when it coincides with the net premium principle.*

Proof. We start from the sufficiency. In the case of $U(x) = ax + b$, with $a > 0$, for any risk X and any insurer's initial capital W, from the equation (1) it follows

$$aW + b = \mathsf{E}[aW + a\pi_{\text{i.e.u.}}[X] - aX + b] = aW + a\pi_{\text{i.e.u.}}[X] - a\mathsf{E}[X] + b,$$

and thus

$$\pi_{\text{i.e.u.}}[X] = \mathsf{E}[X] = \pi_{\text{net}}[X].$$

On the other hand, from the equation (1), for any $\Theta > 0$, the same risk X, the same insurer's initial capital W, and the same insurer's utility function, it follows

$$aW + b = \mathsf{E}[aW + a\pi_{\text{i.e.u.}}[\Theta X] - a\Theta X + b] = aW + a\pi_{\text{i.e.u.}}[\Theta X] - a\Theta\mathsf{E}[X] + b,$$

hence

$$\pi_{\text{i.e.u.}}[\Theta X] = \Theta\mathsf{E}[X] = \Theta\pi_{\text{i.e.u.}}[X],$$

and we see that the scale invariance property holds in this particular case.

The proof of the sufficiency was completed, so we switch to the necessity.

To show that the insurer equivalent utility premium calculation principle with a non-linear insurer's utility function $U(x)$ will not possess the scale invariance property, we will choose a risk X which takes only two possible values, namely, 0 and t (here t is a non-zero real parameter) with probabilities $1 - p$ and p respectively. The risk X can in this case be considered as a random function of two parameters, namely p and t, and, therefore, within the proof of Theorem 5.1 it will be denoted as X_p^t.

For any insurer's initial capital W, and any insurer's utility function $U(\cdot)$, the equivalent utility equation (1) for the risk X_p^t will take a form

$$U(W) = U(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \cdot p + U(W + \pi_{\text{i.e.u.}}[X_p^t]) \cdot (1-p). \tag{95}$$

Substituting $p = 1$ into the equation (95), we get

$$\begin{aligned} U(W) &= U(W + \pi_{\text{i.e.u.}}[X_1^t] - t) \cdot 1 + U(W + \pi_{\text{i.e.u.}}[X_1^t]) \cdot 0 \\ &= U(W + \pi_{\text{i.e.u.}}[X_1^t] - t). \end{aligned} \tag{96}$$

Since $U(x)$ is a strictly increasing function, then the identity (96) yields

$$\pi_{\text{i.e.u.}}[X_1^t] = t. \tag{97}$$

Taking partial derivatives with respect to p from both sides of the equation (95), obtain

$$\begin{aligned} 0 &= U'(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \cdot \frac{\partial}{\partial p} \pi_{\text{i.e.u.}}[X_p^t] \cdot p + U(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \\ &+ U'(W + \pi_{\text{i.e.u.}}[X_p^t]) \cdot \frac{\partial}{\partial p} \pi_{\text{i.e.u.}}[X_p^t] \cdot (1-p) - U(W + \pi_{\text{i.e.u.}}[X_p^t]). \end{aligned} \tag{98}$$

Substituting $p = 1$ into the equation (98), and using the identity (97), we obtain an equation

$$U'(W) \cdot \frac{\partial}{\partial p} \pi_{\text{i.e.u.}}[X_p^t] \Big|_{p=1} = U(W+t) - U(W). \tag{99}$$

Since the premium calculation principle has to be scale invariant, then for any insurer's initial capital W, and any positive constant Θ, the insurer's equivalent utility equation (1) for the risk ΘX_p^t can be written in the following way

$$U(W) = U(W + \Theta\pi_{\text{i.e.u.}}[X_p^t] - \Theta t) \cdot p + U(W + \Theta\pi_{\text{i.e.u.}}[X_p^t]) \cdot (1-p). \tag{100}$$

Calculating partial derivatives with respect to p from both sides of the equation (100), obtain

$$\begin{aligned} 0 &= U'(W + \Theta\pi_{\text{i.e.u.}}[X_p^t] - \Theta t) \cdot \Theta \cdot \frac{\partial}{\partial p} \pi_{\text{i.e.u.}}[X_p^t] \cdot p + U(W + \Theta\pi_{\text{i.e.u.}}[X_p^t] - \Theta t) \\ &+ U'(W + \Theta\pi_{\text{i.e.u.}}[X_p^t]) \cdot \Theta \cdot \frac{\partial}{\partial p} \pi_{\text{i.e.u.}}[X_p^t] \cdot (1-p) - U(W + \Theta\pi_{\text{i.e.u.}}[X_p^t]). \end{aligned} \tag{101}$$

Substituting $p = 1$ into the equation (101), and using the identity (97), we obtain an equation

$$U'(W) \cdot \Theta \cdot \frac{\partial}{\partial p} \pi_{\text{i.e.u.}}[X_p^t] \Big|_{p=1} = U(W + \Theta t) - U(W). \tag{102}$$

Since $\Theta > 0$, then the equation (102) can be rewritten in the following way

$$U'(W) \cdot \frac{\partial}{\partial p} \pi_{\text{i.e.u.}}[X_p^t] \Big|_{p=1} = \frac{U(W + \Theta t) - U(W)}{\Theta}. \tag{103}$$

Note that the equations (99) and (103) have equal left-hand sides, and hens their right-hand sides also have to be equal; this finally gives us an equation which the insurer's utility function has to satisfy for the premium calculation principle to be scale invariant, namely,

$$U(W+t) - U(W) = \frac{U(W + \Theta t) - U(W)}{\Theta}. \tag{104}$$

Taking partial derivatives with respect to the parameter t from both sides of (104) we get

$$U'(W+t) = U'(W + \Theta t). \tag{105}$$

By fixing values of the parameters W and t, and changing values of the parameter Θ, we will make $U'(W + \Theta t)$ a function of changing variable while the value $U'(W+t)$ will be a fixed constant. Using this technique and taking into account monotonicity of the function $U(\cdot)$ and continuity of the function $U'(\cdot)$, since $U(\cdot) \in C_2(\mathbb{R})$, as well as using equation (105) we conclude that

$$U'(x) = a > 0, \quad \text{for} \quad x \in \mathbb{R}.$$

Integration yields

$$U(x) = ax + b, \quad \text{for } x \in \mathbb{R}, \text{ and some constant } a > 0.$$

Let us give also a geometrical interpretation showing that the non-linear insurer's utility functions will not satisfy the equation (104). Let us consider two triangles: the first one will be formed by the points $(W, U(W))$, $(W+t, U(W))$, $(W+t, U(W+t))$ and the second one will be formed by the points $(W, U(W))$, $(W+\Theta t, U(W))$, $(W+\Theta t, U(W+\Theta t))$. Observe that both triangles are right-angled triangles, they have a common vertex at the point $(W, U(W))$, and, moreover, the points $(W, U(W))$, $(W+t, U(W))$, and $(W+\Theta t, U(W))$ lie on the same straight line. With out of loss of generality, equation (104) can be rewritten in the following way

$$\frac{U(W+t) - U(W)}{(W+t) - W} = \frac{U(W+\Theta t) - U(W)}{(W+\Theta t) - W}. \tag{106}$$

Geometrically, equation (106) can be interpreted as follows: ratio of the cathetuses in one of the triangles is equal to the ratio of the cathetuses in the other triangle, hence our two considered triangles are similar triangles. Due to the common vertex, the cathetuses which lie on a common straight line, and the vertexes which lie on the same half-plane with respect to the mentioned line, we conclude that the hypotenuses will also lie on a common straight line; in the other words, the points $(W+\Theta t, U(W+\Theta t))$, for any initial capital W, all non-zero t, and all $\Theta > 0$, will form a straight line. So, we can conclude that the insurer's utility function $U(x)$ is a linear function, i.e., a function of the form $U(x) = ax + b$. Initial assumption of positivity of first derivative of the function $U(x)$ gives us additional restriction on the parameter a: parameter a must be a strictly positive constant. This completes the proof of Theorem 5.1. □

Applying contradiction technique, proof of Theorem 5.1 can be used for showing that the case of $U(x) = ax + b$, for $a > 0$, is the only case when the insurer equivalent utility premium principle coincides with the net premium principle. Indeed, let us assume that for some function $U(x)$, different from the linear function, the insurer equivalent utility principle will be equivalent to the net premium principle. Then, due to the linearity property of the expectation, such method of pricing must be scale invariant. However it was shown in the proof of Theorem 5.1 that the only case when the insurer equivalent utility principle will be scale invariant is the case of $U(x) = ax + b$, for $a > 0$, so we come to a contradiction.

Due to the arbitrary choice of the insurer's initial capital in the proof of Theorem 5.1 and no restrictions on it within the proof, the following useful corollary to Theorem 5.1 can be formulated.

Corollary 5.1. *The insurer zero utility premium calculation principle possesses the scale invariance property if and only if* $U(x) = ax + b$, *for* $a > 0$, *i.e., only in the case when it coincides with the net premium principle.*

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