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Limit Behavior of the Expected Esscher Transform

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Abstract

We present two limit theorems describing convergence of the expected (averaged) Esscher transform to the essential supremum of the variable under transformation when transformation's parameter tends to plus infinity as well as convergence of the same object to the essential infimum of the variable under transformation when transformation's parameter tends to minus infinity.

Keywords: (Expected/averaged) Esscher transform, Esscher premium, Limit behavior, Essential supremum, Essential infimum, Insurance premium.

1. Introduction

Let us consider an arbitrary random variable X with distribution function $F_X(x)$.

Esscher transform, with parameter $\alpha \in R$, of the random variable X is called a random variable $\mathrm{ET}[X;\alpha]$ which is defined as follows

$$\mathrm{ET}[X;\alpha] := \frac{Xe^{\alpha X}}{\mathsf{E}[e^{\alpha X}]}.$$

Expected Esscher transform, with parameter $\alpha \in R$, of the random variable X is defined as the expected value of corresponding Esscher transform, i.e.,

$$\mathrm{EET}[X;\alpha] := \mathsf{E}\big[\mathrm{ET}[X;\alpha]\big] = \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]}.$$

Esscher transform was introduced by the Swedish mathematician Fredrik Esscher in the paper Esscher (1932). The results of the paper Esscher (1932) where later extended and this led to publication of the paper Esscher (1963). Description of know results related to the Esscher transform as well as its applications, in particular to the insurance risk theory, can be found in the reviewing paper by Yang (2004).

To analyze limit behavior of the expected/averaged Esscher transform, we will need to remind the following two definitions:

essential infimum of the random variable X, namely,

$$\operatorname{ess\,inf}[X] := \inf\{\delta : F_X(\delta) > 0\};$$

and *essential supremum* of the random variable X, namely,

$$\text{ess sup}[X] := \sup\{\delta : F_X(\delta) < 1\}.$$

To make our notations a bit more compact, several times within the proofs we will use the following abbreviations:

$$\underline{b} := \text{ess inf}[X] \quad \text{and} \quad \overline{b} := \text{ess sup}[X].$$

2. Convergence to the Essential Supremum

Expected (averaged) Esscher transform possesses the following limit behavior when the transformation parameter α grows to plus infinity.

Theorem 1. For any random variable X, holds limit relation

$$\lim_{\alpha\to+\infty} \text{EET}[X;\alpha] = \text{ess sup}[X]. \quad (1)$$

Proof. For computational convenience, while proving Theorem 1, we will consider separately the following several cases:

Case 1: $\text{ess inf}[X] = \text{ess sup}[X] < +\infty;$

Case 2: $\text{ess inf}[X] < \text{ess sup}[X] \le 0;$

Case 3: $\text{ess inf}[X] < 0 < \text{ess sup}[X] < +\infty;$

Case 4: $0 \le \text{ess inf}[X] < \text{ess sup}[X] < +\infty;$

Case 5: $0 \le \text{ess inf}[X] < \text{ess sup}[X] = +\infty;$

Case 6: $\text{ess inf}[X] < 0, \quad \text{and} \quad \text{ess sup}[X] = +\infty.$

Let us show from the beginning that the limit expected (averaged) Esscher transform for any random variable X will not exceed essential supremum of the random variable X. Observe that it is enough to demonstrate the validity of this statement in the case when $\text{ess sup}[X] < +\infty$ because otherwise the inequality holds automatically. Indeed, here we get

$$\lim_{\alpha\to+\infty} \text{EET}[X;\alpha] = \lim_{\alpha\to+\infty} \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \le \lim_{\alpha\to+\infty} \frac{\mathsf{E}[\text{ess sup}[X]e^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \le$$

$$\le \lim_{\alpha\to+\infty} \frac{\text{ess sup}[X]\mathsf{E}[e^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} = \text{ess sup}[X]. \quad (2)$$

Case 1 corresponds to the situation when the random variable X is a degenerated one, or, in other words, when it is equal to a constant with probability one, namely $P\{X = C\} = 1$ for some $C \in R$. In this case we get

$$\lim_{\alpha\to+\infty} \text{EET}_{\textbf{case 1}}[X;\alpha] = \lim_{\alpha\to+\infty} \frac{Ce^{\alpha C}}{e^{\alpha C}} = C = \text{ess sup}[X],$$

hence, limit relation (1) holds in the first considered case.

Since we analyze the behavior of the stochastic functionals when the transformation parameter $\alpha \in R$ tends to plus infinity, then, without of loss of generality, within the proof of Theorem 1 we may treat the parameter α as a strictly positive one.

In the second, the third, and the fourth considered cases for any ε satisfying inequality

$$0 < \varepsilon < \text{ess sup}[X] - \text{ess inf}[X], \tag{3}$$

holds the following integral limit relation

$$\int_{\underline{b}}^{\bar{b}-\varepsilon} e^{\alpha x} dF_X(x) = o\left(\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)\right) \quad \text{as} \quad \alpha \to +\infty. \tag{4}$$

To prove the integral limit relation (4), we will need the following Markov type inequalities (based on the maximum taken by the integrated function on the interval of integration)

$$\int_{\underline{b}}^{\bar{b}-\varepsilon} e^{\alpha x} dF_X(x) < e^{\alpha(\bar{b}-\varepsilon)} \mathsf{P}\{X \le \bar{b}-\varepsilon\}, \tag{5}$$

and (based on the minimum taken by the integrated function on the interval of integration)

$$\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) \ge \int_{\bar{b}-\varepsilon/2}^{\bar{b}} e^{\alpha x} dF_X(x) > e^{\alpha(\bar{b}-\varepsilon/2)} \mathsf{P}\left\{X \ge \bar{b}-\varepsilon/2\right\} > 0. \tag{6}$$

Using inequalities (5) and (6), obtain

$$0 < \frac{\int_{\underline{b}}^{\bar{b}-\varepsilon} e^{\alpha x} dF_X(x)}{\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)} < \frac{\mathsf{P}\{X \le \bar{b}-\varepsilon\} e^{\alpha(\bar{b}-\varepsilon)}}{e^{\alpha(\bar{b}-\varepsilon/2)} \mathsf{P}\left\{X \ge \bar{b}-\varepsilon/2\right\}} =$$

$$= \frac{\mathsf{P}\{X \le \bar{b}-\varepsilon\}}{\mathsf{P}\left\{X \ge \bar{b}-\varepsilon/2\right\}} e^{-\frac{\alpha\varepsilon}{2}} = \text{constant}\cdot e^{-\frac{\alpha\varepsilon}{2}} \to 0 \quad \text{as} \quad \alpha \to +\infty,$$

Hence, in the mentioned cases the integral limit relation (4) indeed holds.

In the second and the fourth considered cases, for any ε satisfying inequalities (3), the expected (averaged) Esscher transform of the random variable X can be represented in the following way

$$\text{EET}_{\textbf{cases 2 and 4}}[X;\alpha] = \frac{\int_{\underline{b}}^{\bar{b}-\varepsilon} x e^{\alpha x} dF_X(x) + \int_{\bar{b}-\varepsilon}^{\bar{b}} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\bar{b}-\varepsilon} e^{\alpha x} dF_X(x) + \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)}. \tag{7}$$

Let us show that in the second considered case holds limit relation

$$\int_{\underline{b}}^{\bar{b}-\varepsilon} x e^{\alpha x} dF_X(x) = o\left(\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)\right) \quad \text{as} \quad \alpha \to +\infty, \tag{8}$$

i.e., the first summand in the numerator of the representation (7) is asymptotically negligible with respect to the second summand in the denominator of the same representation when the transformation parameter α tends to plus infinity.

Indeed, for a fixed value of the parameter $\alpha > 0$, the integrated function $xe^{\alpha x}$ is decreasing for $x \in (-\infty, -1/\alpha)$, is increasing for $x \in (-1/\alpha, +\infty)$, and is negative for all $x \le \bar{b} - \varepsilon < 0$. For $-1/\alpha \ge \bar{b} - \varepsilon$, i.e., for $\alpha \ge 1/(\varepsilon - \bar{b})$, the first summand in the numerator of the representation (7) can be majorated in the following way

$$\left| \int_{\underline{b}}^{\bar{b}-\varepsilon} xe^{\alpha x} dF_X(x) \right| < \left| (\bar{b} - \varepsilon) e^{\alpha(\bar{b}-\varepsilon)} \mathsf{P}\left\{ X \le \bar{b} - \varepsilon \right\} \right| = (\varepsilon - \bar{b}) e^{\alpha(\bar{b}-\varepsilon)} \mathsf{P}\left\{ X \le \bar{b} - \varepsilon \right\}. \tag{9}$$

Combining inequality (9) with inequality (6), obtain

$$0 < \frac{\left| \int_{\underline{b}}^{\bar{b}-\varepsilon} xe^{\alpha x} dF_X(x) \right|}{\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)} < \frac{(\varepsilon - \bar{b}) e^{\alpha(\bar{b}-\varepsilon)} \mathsf{P}\left\{ X \le \bar{b} - \varepsilon \right\}}{e^{\alpha(\bar{b}-\varepsilon/2)} \mathsf{P}\left\{ X \ge \bar{b} - \varepsilon/2 \right\}} = \text{constant} \cdot e^{-\frac{\alpha\varepsilon}{2}} \to 0 \quad \text{as} \quad \alpha \to +\infty,$$

hence, in the mentioned cases the integral limit relation (8) indeed holds.

Dividing numerator and denominator of the representation (7) by the integral $\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)$ and using the integral limit relations (4) and (8), obtain

$$\begin{aligned} \mathrm{EET}_{\textbf{case 2}}[X;\alpha] &= \frac{\left(\int_{\underline{b}}^{\bar{b}-\varepsilon} xe^{\alpha x} dF_X(x) + \int_{\bar{b}-\varepsilon}^{\bar{b}} xe^{\alpha x} dF_X(x) \right) / \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\bar{b}-\varepsilon} e^{\alpha x} dF_X(x) / \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) + 1} \\ &\sim \int_{\bar{b}-\varepsilon}^{\bar{b}} xe^{\alpha x} dF_X(x) / \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) \quad \text{as} \quad \alpha \to +\infty \\ &\ge (\bar{b} - \varepsilon) \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) / \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) = \bar{b} - \varepsilon. \end{aligned} \tag{10}$$

Combining inequalities (2) and (10), conclude

$$\lim_{\alpha \to +\infty} \mathrm{EET}_{\textbf{case 2}}\left[X;\alpha \right] \in \bigcap_{0 < \varepsilon < \bar{b} - \underline{b}} \left[\bar{b} - \varepsilon, \bar{b} \right] \equiv \left\{ \bar{b} \right\},$$

hence, limit relation (1) holds in the second considered case.

In the fourth considered case, the first summand in the numerator of the representation (7) is non-negative. This allows us to write the following lower bound estimate for the expected (averaged) Esscher transform of the random variable X

$$\mathrm{EET}_{\textbf{case 4}}[X;\alpha] \ge \frac{\int_{\bar{b}-\varepsilon}^{\bar{b}} xe^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\bar{b}-\varepsilon} e^{\alpha x} dF_X(x) + \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)} =: \mathrm{EET}_{\textbf{case 4}}^{\textbf{estimate}}[X;\alpha,\varepsilon].$$

Dividing numerator and denominator of the lower bound estimate $\mathrm{EET}_{\textbf{case 4}}^{\textbf{estimate}}[X;\alpha,\varepsilon]$ by the integral $\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)$, switching to the limit when α grows to plus infinity, and using the integral limit relation (4), obtain

$$\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 4}}[X;\alpha,\varepsilon] \;=\; \frac{\int_{\bar b-\varepsilon}^{\bar b} xe^{\alpha x}dF_X(x) \,/\, \int_{\bar b-\varepsilon}^{\bar b} e^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{\bar b-\varepsilon} e^{\alpha x}dF_X(x) \,/\, \int_{\bar b-\varepsilon}^{\bar b} e^{\alpha x}dF_X(x)+1}$$

$$\sim\; \int_{\bar b-\varepsilon}^{\bar b} xe^{\alpha x}dF_X(x) \,/\, \int_{\bar b-\varepsilon}^{\bar b} e^{\alpha x}dF_X(x) \quad \text{as} \quad \alpha\to+\infty$$

$$\ge\; (\bar b-\varepsilon)\int_{\bar b-\varepsilon}^{\bar b} e^{\alpha x}dF_X(x) \,/\, \int_{\bar b-\varepsilon}^{\bar b} e^{\alpha x}dF_X(x) \;=\; \bar b-\varepsilon,$$

Hence, in the fourth considered case

$$\lim_{\alpha\to+\infty}\mathrm{EET}_{\mathbf{case\ 4}}[X;\alpha] \;\ge\; \lim_{\alpha\to+\infty}\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 4}}[X;\alpha,\varepsilon] \;\ge\; \bar b-\varepsilon. \tag{11}$$

Combining inequality (11) with inequalities (2) and (3), we conclude that

$$\lim_{\alpha\to+\infty}\mathrm{EET}_{\mathbf{case\ 4}}\left[X;\alpha\right] \;\in\; \bigcap_{0<\varepsilon<\bar b-\underline{b}}\left[\bar b-\varepsilon,\bar b\right] \;\equiv\; \left\{\bar b\right\},$$

Hence, asymptotic relation (1) holds also in the fourth considered case.

In the third considered case, for any ε satisfying inequalities

$$0<\varepsilon<\operatorname{ess\,sup}[X], \tag{12}$$

The expected (averaged) Esscher transform of the random variable X can be represented in the following way

$$\mathrm{EET}_{\mathbf{case\ 3}}[X;\alpha] \;=\; \frac{\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x)+\int_{0}^{\bar b-\varepsilon} xe^{\alpha x}dF_X(x)+\int_{\bar b-\varepsilon}^{\bar b} xe^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{\bar b-\varepsilon} e^{\alpha x}dF_X(x)+\int_{\bar b-\varepsilon}^{\bar b} e^{\alpha x}dF_X(x)}$$

$$\ge\; \frac{\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x)+\int_{\bar b-\varepsilon}^{\bar b} xe^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{\bar b-\varepsilon} e^{\alpha x}dF_X(x)+\int_{\bar b-\varepsilon}^{\bar b} e^{\alpha x}dF_X(x)} \;=:\; \mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 3}}[X;\alpha,\varepsilon],$$

Where the last inequality holds due to the non-negativity of the integral $\int_{0}^{\bar b-\varepsilon} xe^{\alpha x}dF_X(x).$

Let us now show that in the third considered case, for any ε satisfying inequalities (12) hold the following integral limit relation

$$\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x) \;=\; o\!\left(\int_{\bar b-\varepsilon}^{\bar b} e^{\alpha x}dF_X(x)\right) \quad \text{as} \quad \alpha\to+\infty, \tag{13}$$

i.e., the first summand in the numerator of the lower bound estimate $\mathrm{EET}^{\mathbf{estimate}}_{\mathbf{case\ 3}}[X;\alpha,\varepsilon]$ is asymptotically negligible with respect to the second summand in the denominator of the same estimate when the transformation parameter α tends to plus infinity.

As was already mentioned, the function $xe^{\alpha x}$ for the fixed positive values of the transformation parameter α takes negative values on the negative half-line and achieves its minimum at the point $x=-1/\alpha,$ hence

$$\left|\int_{\underline{b}}^{0} xe^{\alpha x} dF_X(x)\right| \;\le\; \left|(-1/\alpha)e^{\alpha(-1/\alpha)}\mathsf{P}\{X \le 0\}\right| \;=\; (1/\alpha)e^{-1}\mathsf{P}\{X \le 0\}. \tag{14}$$

Note that in the third considered case, ε satisfying inequalities (12) will also satisfy inequalities (3). Combining inequality (14) with inequality (6), obtain

$$0 \;<\; \frac{\left|\int_{\underline{b}}^{0} xe^{\alpha x} dF_X(x)\right|}{\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)} \;<\; \frac{(1/\alpha)e^{-1}\mathsf{P}\{X \le 0\}}{e^{\alpha(\bar{b}-\varepsilon/2)}\mathsf{P}\{X \ge \bar{b}-\varepsilon/2\}} \;=\; \text{constant}/\alpha\cdot e^{-\alpha(\bar{b}-\varepsilon/2)} \to 0 \quad \text{as} \quad \alpha \to +\infty,$$

thus, in the third considered case, the integral limit relation (13) indeed holds.

Dividing numerator and denominator of the lower bound estimate $\mathrm{EET}_{\mathbf{case\ 3}}^{\mathbf{estimate}}[X;\alpha,\varepsilon]$ by the integral $\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)$, switching to the limit as the transformation parameter α tends to plus infinity, and using the integral limit relations (4) and (13), obtain

$$\begin{aligned}\mathrm{EET}_{\mathbf{case\ 3}}^{\mathbf{estimate}}[X;\alpha,\varepsilon] \;&=\; \frac{\left(\int_{\underline{b}}^{0} xe^{\alpha x} dF_X(x) + \int_{\bar{b}-\varepsilon}^{\bar{b}} xe^{\alpha x} dF_X(x)\right) / \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{\bar{b}-\varepsilon} e^{\alpha x} dF_X(x) / \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) + 1} \\ &\sim \int_{\bar{b}-\varepsilon}^{\bar{b}} xe^{\alpha x} dF_X(x) / \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) \quad \text{as} \quad \alpha \to +\infty \\ &\ge (\bar{b}-\varepsilon)\int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) / \int_{\bar{b}-\varepsilon}^{\bar{b}} e^{\alpha x} dF_X(x) \;=\; \bar{b}-\varepsilon.\end{aligned} \tag{15}$$

From the limit inequality (15), for any ε satisfying (12), it follows

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 3}}[X;\alpha] \;\ge\; \lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 3}}^{\mathbf{estimate}}[X;\alpha,\varepsilon] \;\ge\; \bar{b}-\varepsilon. \tag{16}$$

Combining inequalities (2), (16), and (12), we conclude

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 3}}[X;\alpha] \;\in\; \bigcap_{0<\varepsilon<\bar{b}} \left[\bar{b}-\varepsilon, \bar{b}\right] \;\equiv\; \{\bar{b}\},$$

hence, asymptotic relation (1) holds also in the third considered case.

In the fifth, and the sixth considered cases for any real constant c satisfying inequalities

$$\max\{0, \text{ess inf}[X]\} < c < +\infty, \tag{17}$$

holds the following integral limit relation

$$\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) \;=\; o\left(\int_{c}^{+\infty} e^{\alpha x} dF_X(x)\right) \quad \text{as} \quad \alpha \to +\infty. \tag{18}$$

Indeed, due to monotonicity of the function $e^{\alpha x}$, holds the following Markov type inequality (based on the maximum taken by the integrated function over the interval of integration)

$$\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) \;<\; e^{\alpha c}\mathsf{P}\{X \le c\}. \tag{19}$$

On the other hand we have the following Markov type inequality (based on the minimum taken by the integrated function over the interval of integration)

$$\int_{c}^{+\infty} e^{\alpha x} dF_X(x) \;\ge\; \int_{2c}^{+\infty} e^{\alpha x} dF_X(x) \;>\; e^{2\alpha c}\mathsf{P}\{X \ge 2c\} \;>\; 0. \tag{20}$$

Using inequalities (19) and (20), we can write

$$0 \;\le\; \frac{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x)}{\int_{c}^{+\infty} e^{\alpha x} dF_X(x)} \;<\; \frac{e^{\alpha c}\mathsf{P}\{X \le c\}}{e^{2\alpha c}\mathsf{P}\{X \ge 2c\}} \;=\; \text{constant}\cdot e^{-\alpha c} \to 0 \quad \text{as} \quad \alpha \to +\infty,$$

hence, in the considered case the integral limit relation (18) indeed holds.

In the fifth considered case the expected (averaged) Esscher transform of the random variable X, for any $c > \underline{b}$, can be represented as

$$\mathrm{EET}_{\textbf{case 5}}[X;\alpha] \;=\; \frac{\int_{\underline{b}}^{c} xe^{\alpha x} dF_X(x) + \int_{c}^{+\infty} xe^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) + \int_{c}^{+\infty} e^{\alpha x} dF_X(x)} \;\ge\; \frac{\int_{c}^{+\infty} xe^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) + \int_{c}^{+\infty} e^{\alpha x} dF_X(x)} \;=:\; \mathrm{EET}_{\textbf{case 5}}^{\textbf{estimate}}[X;\alpha,c],$$

Where the last inequality holds due to the non-negativity of the integral $\int_{\underline{b}}^{c} xe^{\alpha x} dF_X(x)$.

Dividing numerator and denominator of the lower bound estimate $\mathrm{EET}_{\textbf{case 5}}^{\textbf{estimate}}[X;\alpha,c]$ by the integral $\int_{c}^{+\infty} e^{\alpha x} dF_X(x)$, switching to the limit as the transformation parameter α tends to plus infinity, and using the integral limit relation (18), obtain

$$\begin{aligned}\mathrm{EET}_{\textbf{case 5}}^{\textbf{estimate}}[X;\alpha,c] \;&=\; \frac{\int_{c}^{+\infty} xe^{\alpha x} dF_X(x) / \int_{c}^{+\infty} e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) / \int_{c}^{+\infty} e^{\alpha x} dF_X(x) + 1} \\ &\sim \int_{c}^{+\infty} xe^{\alpha x} dF_X(x) / \int_{c}^{+\infty} e^{\alpha x} dF_X(x) \quad \text{as} \quad \alpha \to +\infty \\ &\ge c\int_{c}^{+\infty} e^{\alpha x} dF_X(x) / \int_{c}^{+\infty} e^{\alpha x} dF_X(x) = c.\end{aligned} \tag{21}$$

From the limit inequality (21) it follows, for any $c > \underline{b}$,

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\textbf{case 5}}[X;\alpha] \;\ge\; \lim_{\alpha\to+\infty} \mathrm{EET}_{\textbf{case 5}}^{\textbf{estimate}}[X;\alpha,c] \;\ge\; c,$$

and therefore

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 5}}\left[X;\alpha\right] \in \bigcap_{c>\underline{b}}\left[c,+\infty\right] \equiv \left\{+\infty\right\},$$

Hence, asymptotic relation (1) holds also in the fifth considered case.

In the sixth considered case, for any real c satisfying inequalities

$$0 < c < +\infty, \tag{22}$$

observe that in the sixth case c satisfying inequalities (22) will also satisfy inequalities (17)) the expected (averaged) Esscher transform can be represented in the following way

$$\mathrm{EET}_{\mathbf{case\ 6}}[X;\alpha] = \frac{\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x) + \int_0^c xe^{\alpha x}dF_X(x) + \int_c^{+\infty} xe^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x}dF_X(x) + \int_c^{+\infty} e^{\alpha x}dF_X(x)}$$

$$\geq \frac{\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x) + \int_c^{+\infty} xe^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x}dF_X(x) + \int_c^{+\infty} e^{\alpha x}dF_X(x)} =: \mathrm{EET}_{\mathbf{case\ 6}}^{\mathbf{estimate}}[X;\alpha,c],$$

Where the last inequality is valid due to the non-negativity of the integral $\int_0^c xe^{\alpha x}dF_X(x).$

Let us now show that in the sixth considered case the following integral limit relation holds

$$\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x) = o\left(\int_c^{+\infty} e^{\alpha x}dF_X(x)\right) \quad \text{as} \quad \alpha\to+\infty, \tag{23}$$

i.e., the first summand in the numerator of the lower bound estimate $\mathrm{EET}_{\mathbf{case\ 6}}^{\mathbf{estimate}}[X;\alpha,c]$ is asymptotically negligible with respect to the second summand in the denominator of the same estimate when the perturbation parameter α tends to plus infinity.

Indeed, combining inequalities (14) and (20), obtain

$$0 \leq \frac{\left|\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x)\right|}{\int_c^{+\infty} e^{\alpha x}dF_X(x)} < \frac{(1/\alpha)e^{-1}\mathsf{P}\left\{X\leq 0\right\}}{e^{2\alpha c}\mathsf{P}\{X\geq 2c\}} = \text{constant}/\alpha\cdot e^{-\alpha c} \to 0 \quad \text{as} \quad \alpha\to+\infty,$$

Hence, in the considered case the integral limit relation (23) indeed holds.

Dividing numerator and denominator of the lower bound estimate $\mathrm{EET}_{\mathbf{case\ 6}}^{\mathbf{estimate}}[X;\alpha,c]$ by the integral $\int_c^{+\infty} e^{\alpha x}dF_X(x),$ switching to the limit as the transformation parameter α tends to plus infinity, and using the integral limit relations (18) and (23), obtain

$$\mathrm{EET}_{\mathbf{case\ 6}}^{\mathbf{estimate}}[X;\alpha,c] \;=\; \frac{\left(\int_{\underline{b}}^{0} xe^{\alpha x}dF_X(x) + \int_{c}^{+\infty} xe^{\alpha x}dF_X(x)\right) / \int_{c}^{+\infty} e^{\alpha x}dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x}dF_X(x) / \int_{c}^{+\infty} e^{\alpha x}dF_X(x) + 1}$$

$$\sim \int_{c}^{+\infty} xe^{\alpha x}dF_X(x) / \int_{c}^{+\infty} e^{\alpha x}dF_X(x) \quad \text{as} \quad \alpha \to +\infty$$

$$\geq\; c \int_{c}^{+\infty} e^{\alpha x}dF_X(x) / \int_{c}^{+\infty} e^{\alpha x}dF_X(x) \;=\; c. \tag{24}$$

From inequality (24), for any c satisfying (22), it follows

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 6}}[X;\alpha] \;\geq\; \lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 6}}^{\mathbf{estimate}}[X;\alpha,c] \;\geq\; c,$$

and therefore

$$\lim_{\alpha\to+\infty} \mathrm{EET}_{\mathbf{case\ 6}}\left[X;\alpha\right] \;\in\; \bigcap_{c>0}\left[c,+\infty\right] \;\equiv\; \left\{+\infty\right\},$$

hence, asymptotic relation (1) holds also in the sixth considered case.

This completes the proof of the theorem.

□

3. Convergence to the Essential Infimum

Expected (averaged) Esscher transform possesses the following limit behavior when transformation parameter α tends to minus infinity.

Theorem 2. For any random variable X holds limit relation

$$\lim_{\alpha\to-\infty} \mathrm{EET}[\mathrm{X};\alpha] = \mathrm{ess\ inf}[\mathrm{X}]. \tag{25}$$

Proof. Using statement of Theorem 1, obtain

$$\lim_{\alpha\to-\infty} \mathrm{EET}[\mathrm{X};\alpha] \;=\; \lim_{\alpha\to-\infty} \frac{\mathsf{E}[\mathrm{X}e^{\alpha \mathrm{X}}]}{\mathsf{E}[e^{\alpha \mathrm{X}}]} \;=\; -\lim_{\alpha\to-\infty} \frac{\mathsf{E}[-\mathrm{X}e^{-\alpha(-\mathrm{X})}]}{\mathsf{E}[e^{-\alpha(-\mathrm{X})}]} \;=\; -\lim_{\beta\to+\infty} \frac{\mathsf{E}[-Xe^{\beta(-X)}]}{\mathsf{E}[e^{\beta(-X)}]} \;=$$

$$=\; -\lim_{\beta\to+\infty} \mathrm{EET}[-X;\beta] \;=\; -\mathrm{ess\ sup}[-X] \;=\; \mathrm{ess\ inf}[X].$$

Hence Theorem 2 can be viewed as a corollary to Theorem 1.

□

Remark 1. Note that for any random variable X and for any $\alpha \in R$ hold inequalities

$$\mathrm{ess\ inf}[X] \;\leq\; \mathrm{EET}[\mathrm{X};\alpha] \;\leq\; \mathrm{ess\ sup}[\mathrm{X}].$$

This means that the expected (averaged) Esscher transform will converge to the essential infimum of X from above as the transformation parameter α tends to minus infinity, and will converge to the essential supremum of X from below as the transformation parameter α tends to plus infinity.

Remark 2. Theorems 1 and Theorem 2 are equivalent to each other in the sense that one could first prove Theorem 2 and then obtain a compact proof of Theorem 1 in a form of corollary to Theorem 2.

Remark 3. Expected (averaged) Esscher transform is widely used in actuarial mathematics as a method of pricing of insurance contracts. There it is known under the name *Esscher premium*. Note that actuaries mainly restrict their attention to the Esscher transform of the non-negative random variable X (which in this case represents a size of the insurance compensation related to some insurance contract) with non-negative values of the parameter α. Application of the expected (averaged) Esscher transform to pricing of the insurance contracts probably was initiated by the Swiss mathematician Hans Bühlmann in the paper Bühlmann (1980).

Some alternative aspects of the results presented above can be found in the paper by Pratsiovytyi and Drozdenko (2016).

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